A resolution calculus for modal logic S4

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1. Introduction

In this paper, we present a resolution calculus for the first-order modal logic S4. The formulas are given not necessary in a clausal form. This method can be used for automatizable proof procedure of a quantified modal logic. We will consider formulas for which the following conditions hold:
1. the formulas $F$ contain only logical connectives $\neg$, $\&$, $\lor$, and no logical or modal symbol in $F$ lies in the scope of a negation,
2. the formulas are closed, i.e., we consider the formulas without free variables,
3. the formulas are transformed into Skolem normal form (see [1],[2]),
4. the formulas are of the form $G_1 \lor G_2 \lor \ldots \lor G_s$, where $G_i$ is a literal or a formula beginning with $\square$, $\Diamond$.

The order of formulas is not fixed in a disjunction or in a conjunction. In what follows, $P, P_1, P_2$ denote the atomic formulas. Formulas are denoted by $F, G, K, H$ and $M$. Moreover, $H$ and $M$ can be the empty formulas as well. The symbol $\perp$ denotes an empty formula.

2. The resolution rules

2.1. Classical rules

\[
(c1) \quad \frac{[P_1 \lor H, \neg P_2 \lor M] \theta}{[H \lor M] \theta}
\]

$\theta$ is an most general unifier of $\{P_1, P_2\}$. We assume that the formulas written over the line have no common individual variables (this if necessary can be obtained by renaming variables). Substitution $\theta$ is a finite set of the form $t_1/x_1, \ldots, t_n/x_n$, where every $x_i$ is a variable, every $t_i$ is a term, different from $x_i$, and for all $i, j$ such that $i \neq j$, $x_i$ differs
from $x_j$. Moreover, if the level (see [1]) of $x$ is $n$ and if the term $t$ contains some symbol whose level is greater than $n$, then the substitution of $t$ for $x$ is forbidden.

\[
\begin{align*}
(c2) & \quad \frac{(F \& G) \lor H}{F \lor H} & (c3) & \quad \frac{\text{res}(P, \neg P)}{\bot} \\
(c4) & \quad \frac{\text{res}(F \lor K, G)}{\text{res}(F, G) \lor K} & (c5) & \quad \frac{\text{res}(F \& K, G)}{K \& \text{res}(F, G)} \\
(c6) & \quad \frac{\text{res}(F \lor G)}{G \lor \text{res}F} & (c7) & \quad \frac{\text{res}(F \& G)}{\text{res}(F, G)} \\
& \quad \frac{\text{res}(F \& G)}{G \& \text{res}F}
\end{align*}
\]

2.2. Modal rules

\[
\begin{align*}
(m1) & \quad \frac{[H \lor \Box F, M \lor \Box G] \theta}{[H \lor M \lor \Box \text{res}(F, G)] \theta} & (m2) & \quad \frac{[H \lor \Box F, M \lor \Diamond G] \theta}{[H \lor M \lor \Diamond \text{res}(F, G)] \theta} \\
(m3) & \quad \frac{[H \lor \Box F] \theta}{[H \lor \Box \text{res}F] \theta} & (m4) & \quad \frac{[H \lor \Diamond F] \theta}{[H \lor \Diamond \text{res}F] \theta} \\
(m5) & \quad \frac{\text{res}(\Box F, \Box H)}{\Box \text{res}(F, H)} & (m6) & \quad \frac{\text{res}(\Box H, \Diamond F)}{\Box \text{res}(H, F)} \\
(m7) & \quad \frac{\text{res}(\Box F, H)}{\text{res}(\Box^{+}, H)} & (m8) & \quad \frac{\text{res}(\Box F, H)}{\text{res}(\Box^{+}, H)} \\
(m9) & \quad \frac{[H \lor \Box F, K] \theta}{[H \lor \text{res}(\Box^{+}, K)] \theta} & (m10) & \quad \frac{[H \lor \Box F, K] \theta}{[H \lor \text{res}(\Box^{+}, K)] \theta}
\end{align*}
\]

$F^-$ is obtained from $F$ (see [1]) by subtracting one from the level of those symbols that have a level greater than the modal degree of $\Box F$.

$F^+$ is obtained from $F$ by adding one to the level of those symbols whose level is greater than the modal degree of $\Box F$.

2.3. Simplification rules

\[
\begin{align*}
(s1) & \quad \frac{F \lor \bot}{F} & (s2) & \quad \frac{F \& \bot}{\bot} & (s3) & \quad \Box \bot \\
(s4) & \quad \frac{\Diamond \bot}{\bot} & (s5) & \quad \frac{\text{res}(\bot, H)}{\bot} & (s6) & \quad \frac{\text{res}(\bot \lor F, H)}{\text{res}(F, H)} \\
(s7) & \quad \frac{\text{res}(\bot \& F, H)}{\bot} & (s8) & \quad \frac{\text{res}(\Box \bot, H)}{\bot} & (s9) & \quad \frac{\Diamond \bot, H}{\bot}
\end{align*}
\]
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2.4. Duplication rule

\[(d1) \quad \frac{F(x^n)}{F(x^n) \& F(y^n)}.\]

Here \(y\) is a new variable, \(x^n\) occurs only in \(F(x^n)\), \(F(x^n)\) is not in the scope of more than \(n\) modal, and \(F(x^n)\) is not in the scope of a negation.

2.5. Factorization rule

\[(f1) \quad \frac{F \vee F \vee H}{F \vee H}.\]

The main results

We define the generalized formulas as follows:

1. If \(F\) is a formula, then \(\text{res}F\) is a generalized formula.
2. If \(F\) and \(G\) are formulas, then \(\text{res}(F, G)\) is a generalized formula.
3. If \(F\) is a generalized formula, then \(\neg F\) is also a generalized formula.
4. If \(F\) is a formula and \(G\) is a generalized formula, then \((F \lor G), (F \land G), (F \rightarrow G), (G \rightarrow F), \Box G, \Diamond G\) are generalized formulas.

Note that we consider only Skolemized formulas. The formulas \(F, G, K, H\) met in the resolution rules do not contain \(\text{res}\).

A derivation of the formula (generalized formula) \(F\) from a set of formulas \(S\) is a finite sequence \(G_1, G_2, \ldots, G_s\) such that

1. \(G_s = F\).
2. \(G_i\) is a formula or a generalized formula.
3. For every \(i \leq s\) at least one of the following conditions holds:
   (a) \(G_i \in S\).
   (b) For some \(j, k < i\) \(F_i\) follows from \(G_j, G_k\) by one of the rules \((c1), (c2), (m1)-(m4), (m9), (m10)\) or \((s1)-(s4)\).
   (c) For some \(j (j < i) G_j = G(\text{res}K)\), i.e., \(\text{res}K\) is a generalized subformula of \(G, G_i = G(\text{res}H)\) (or \(G_i = G(H)\)) and \(\text{res}H\) (or \(H\)) follows from \(\text{res}K\) by one of the rules \((c3)-(c8), (m5)-(m8)\) or \((s5)-(s9)\).
   (d) For some \(j G_j = G(F(x^n))\) and \(G_i = G(F(x^n) \& F(y^n))\). Here \(y\) is a new variable satisfying the conditions of the rule \((d1)\).
   (e) For some \(j < i G_j = G(K)\) is a formula, \(G_j = G(M)\) and \(M\) follows from \(K\) by one of the rules \((s1)-(s4)\) or \((f1)\).

Theorem 1. \(S \vdash \bot\) if and only if \(S\) is refutable.

Proof. Soundness and completeness of a resolution modal system S4 is proved in [1]. We will show that every application of a rule of resolution modal system in [1] is simulated by a finite sequence of applications of considered calculus.
Assume that a formula which does not satisfy the Condition 4 described in the introduction is obtained. In this case, we can obtain the required form by applying a finite number of rule (c2).

Each application of rules (m1)–(m4), (m9) and (m10) introduces generalized formulas containing res. The rules (c3)–(c8), (m5)–(m8), (s5)–(s9) and (d1) present recursive transformation of generalized formulas, i.e., of the formulas containing res. We simulate the applications of the rules (c1), (c2), (m1)–(m4), (m9) and (m10) for the subformulas which are in the scope of res using the above-introduced resolution rules. As a result a simplified formula not containing res can be obtained by applying the rules (s5)–(s9).

The rule (c2) from [1] of the form if \( C \) is a \( \theta \)-resolvent of \( S' \cup \{ A \} \), then \( C \lor B \theta \) is a \( \theta \)-resolvent of \( S' \cup \{ A \lor B \} \) is simulated by rules (c1), (c4), (c6) of the calculus in question.

Rule (c3) from [1] of the form if \( C \) is a \( \theta \)-resolvent of \( S' \cup \{ A \} \), then \( C \land B \theta \) is a \( \theta \)-resolvent of \( S' \cup \{ A \land B \} \) is simulated by rules (c5) and (c8) of a considered calculus.

Rule (c4) from [1] of the form if \( C \) is a \( \theta \)-resolvent of \( \{ A, B \} \), then \( C \land \theta \lor A \land B \) is simulated by rule (c7) of a considered calculus.

Rules (m1)–(m4) from [1] are simulated by the corresponding rules (m2), (m3), (m1) and (m4) of a considered calculus.

The simplifications rules from [1] are simulated by rules (s1)–(s9) of a respective calculus. Moreover, each formula of a considered calculus is a particular case of some rule from [1]. The theorem is proved.

Consider now the formulas of propositional modal logic for which the following conditions hold:

- the formulas \( F \) contain only logical connectives \( \neg \) and \( \lor \),
- no logical or modal symbol lies in the scope of a negation.

Now, we shall present our calculus in this particular case (\( p \) denotes a propositional variable).

**Calculus MS4**

\[
\begin{align*}
\text{(c1)} & \quad \frac{p \lor H, \neg p \lor M}{H \lor M} \\
\text{(m1)} & \quad \frac{H \lor \Box p, \neg p \lor M}{H \lor M} \\
\text{(m3)} & \quad \frac{H \lor \Box F, M \lor \Box G}{H \lor M \lor \Box res(F,G)} \\
\text{(m5)} & \quad \frac{res(H \lor \Box p, \neg p \lor M)}{H \lor M} \\
\text{(m7)} & \quad \frac{res(H \lor \Box F, M \lor \Box G)}{H \lor M \lor \Box res(F,G)} \\
\text{(c2)} & \quad \frac{res(p \lor H, \neg p \lor M)}{H \lor M} \\
\text{(m2)} & \quad \frac{H \lor \Box p, \neg p \lor M}{H \lor M} \\
\text{(m4)} & \quad \frac{H \lor \Box F, M \lor \Box G}{H \lor M \lor \Box res(F,G)} \\
\text{(m6)} & \quad \frac{res(H \lor \Box p, \neg p \lor M)}{H \lor M} \\
\text{(m8)} & \quad \frac{res(H \lor \Box F, M \lor \Box G)}{H \lor M \lor \Box res(F,G)}
\end{align*}
\]
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\[
\begin{align*}
(s1) & \quad \Box F \quad (s2) \quad \Box F \quad (s3) \quad \Box \bot \\
(s4) & \quad \Diamond F \quad (s5) \quad F \lor \bot \quad (f1) \quad F \lor F \lor H
\end{align*}
\]

DEFINITION 1. A derivation of a formula \( F \) from the set of formulas \( S \) is a finite sequence \( G_1, G_2, \ldots, G_s \) such that
1. \( G_i (i = 1, 2, \ldots, s) \) is a formula or a generalized formula.
2. \( G_s = F \).
3. For every \( i \leq s \) at least one of the following conditions holds:
   (a) \( G_i \in S \),
   (b) For some \( j \) and \( k < i \) \( F \) follows from \( G_j \) and \( G_k \) by one of the rules \((c1)\)–\((c4)\).
   (m1)–(m4).
   (c) For some \( j < i \) \( G_j = G(resK) \), i.e., \( resK \) is a generalized subformula of \( G \), \( G_i = G(H) \) and \( H \) follows from \( resK \) by one of rules \((c2)\)–\((c4)\), \((m5)\)–\((m8)\).
   (d) For some \( j < i \) \( G_j = G(K) \) (\( K \) does not contain \( res \)), \( G_i = G(H) \) and \( H \) follows from \( K \) by one of the rules \((s3)\)–\((s5)\), \((f1)\).

Disjunctions of modal literals are called modal clauses. Modal literals are expressions of the form \( q, \Box q \) or \( \Diamond q \), where \( q \) is a propositional variable or its negation. Initial modal clauses are expressions of the form \( \Box C \), where \( C \) is a modal clause. The following proposition is improved in \([3]\): for any formula \( F \) one can construct (by introduction of new variables) the list \( X_p \) of initial clauses and a propositional variable \( g \) such that \( \vdash S4 F \) if and only if \( \vdash S4 \Box X_F \rightarrow g \).

It means that, in the general case, we can consider the set \( S \) of input formulas containing only modal and initial clauses. Note that the rules of MS4 allow us to derive from \( S \) formulas which are not initial (or modal) clauses.

For example, \( \Box \neg p \lor \Box q, \Box (r \lor \neg q \lor \neg s) \vdash_{MS4} \Box \neg p \lor \Box (r \lor \neg s) \).

Literatūra